Matrix Methods in the Calculation and Analysis of Orbits

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Formulas are presented for a method for the calculation of special perturbations using matrix methods that had been formerly adapted to problems of general perturbations The practical application of these formulas is described (They were programmed and tested with a circumlunar orbit) Once calculated, the results for the orbit are in such a form that its analysis (error analysis, guidance, and so on) can be carried out with the use of simple closed functions; no further integration is necessary

1 Introduction

IN 1962 the author published a paper[1] dealing with the application of matrix methods to the calculation of general perturbations in celestial mechanics In the paper some rather general remarks were made concerning the possibility of applying the basic formula to the calculation of special perturbations The author was fortunate to receive support from Minneapolis-Honeywell Regulator Company to look into this further, and as a result the method has been developed and tested This paper contains a summary of the method and of the basic formulas

As a somewhat acid test, the method was tried for the case of a circumlunar orbit leaving and later returning to the earth, while passing well within the moon's sphere of influence It was found to be quite successful and quite speedy, if an over-all accuracy of not less than a mile was required But one of its principal attractions is that it leaves the calculated orbit in such a state that the analysis of the orbit can be performed entirely by the use of closed analytical formulas; no further integration is necessary

The type of analysis considered here might involve the prediction of the range of error resulting from some uncertainty in initial conditions, or the calculation of small artificial changes in velocity designed to produce a modification at a later stage in an orbit, or the calculation of small changes to initial conditions needed when searching for a suitable orbit for some mission, etc , etc It is necessary that some *reference orbit* be known, and any deviations from it must be assumed to be so small that their squares and products are negligible This makes possible a first-order, linear analysis To consider the limitations of this in numerical terms, suppose that six places of decimals are required in a calculation Then certainly no departure from the reference orbit involving numbers greater than 10^{-3} should be allowed This limitation has a compensating factor, for the expressions used in the analysis need not be as accurate as those used in the acquisition of the reference orbit itself For instance, no planet moves in a Keplerian orbit; but formulas pertaining to Keplerian motion can be used in some problems concerning differential corrections of planetary orbits An advantage of this is that these formulas are simple and well-known

But if an orbit is nowhere near Keplerian, no relevant analytical expressions are likely to be known The figures for analyzing the orbit may be found by solving the equations of motion repeatedly subject to slightly varying initial conditions, or they may be found by solving the first variational system of the equations of motion (and perhaps also the adjoint system) Either process involves a deal of numerical integration But it may be possible for the orbit to be divided into arcs, each of which adequately resembles a Keplerian orbit, and in this case the analysis of the whole can be made using the explicitly known formulas for the parts This is even so if the separate Keplerian arcs have different origins, as is the case with a circumlunar orbit

To explain the use of the word "adequately" we revert to the foregoing numerical illustration The greatest deviation from the reference Keplerian orbit to the *true orbit* should not exceed 10^{-3} when put into numerical terms This could lead to an absurd degree of subdivision But it was found for the circumlunar orbit that eight arcs were adequate

Given any integrated orbit, it would not be impossible to divide it into suitable arcs This division is automatic if the orbit is calculated using the methods described below, and will therefore be approached in this way

2 The Principles of the Method

Suppose that the motion of a particle is subject to a dominant inverse-square-law field of force $-\mu\mathbf{r}/r^3$ and an additional disturbing force $\mathbf{f}(\mathbf{r},t)$ Let some *reference* orbit be chosen that satisfies the equations of motion with $\mathbf{f} = 0$, and adequately resembles the *true* orbit over some interval of time (the choice is described below) Let small *residuals*, or *perturbations* $\delta\mathbf{r}$ and $\delta\mathbf{r}'$, in position and velocity, respectively, be added to the corresponding quantities in the reference orbit at time t to attain the true orbit

If $\mathbf{f}$ were zero, then the relation between residuals at times t_0 and t would be of the type

$$\begin{bmatrix}\delta\mathbf{r}\\ \delta\mathbf{r}'\end{bmatrix} = \mathbf{\Omega}(t,t_0)\begin{bmatrix}\delta\mathbf{r}_0\\ \delta\mathbf{r}_0'\end{bmatrix} \tag{1}$$

where the vector $\delta\mathbf{r}$ represents the matrix

$$\begin{bmatrix}\delta x\\ \delta y\\ \delta z\end{bmatrix}, \text{ etc}$$

Axes will be taken as Cartesian and nonrotating $\mathbf{\Omega}(t,t_0)$ is the *matrizant* of Keplerian motion; it is documented elsewhere [2] It is convenient to subdivide it into four three-by-three matrices, and the formulas are simplified if the following property is used:

$$\begin{aligned}\mathbf{\Omega}(t,t_0) &= \begin{bmatrix}\mathbf{L}(t,t_0) & \mathbf{M}(t,t_0)\\ \mathbf{P}(t,t_0) & \mathbf{Q}(t,t_0)\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{L}(t,T) & \mathbf{M}(t,T)\\ \mathbf{P}(t,T) & \mathbf{Q}(t,T)\end{bmatrix}\begin{bmatrix}\mathbf{Q}^{\mathrm{T}}(t_0,T) & -\mathbf{M}^{\mathrm{T}}(t_0,T)\\ -\mathbf{P}^{\mathrm{T}}(t_0,T) & \mathbf{L}^{\mathrm{T}}(t_0,T)\end{bmatrix}\end{aligned} \tag{2}$$

Here a superscript T means that the matrix is transposed The parameter T is arbitrary in (2), but to simplify the formulas it will be taken to be a time of pericentron passage

Received May 28, 1963; revision received October 16, 1963 This work was supported by Minneapolis-Honeywell Regulator Company, Aeronautical Division I am greatly in their debt, for without that support the method would not have been cast into a workable form I am also most grateful to J G Hartwell for his work in the programming of the method

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Table 1 Elliptic reference orbit

$$
\begin{aligned}
C &= \cos E \qquad S = \sin E \\
\tilde{l}_{11} &= (1 + e - e^2)C^2 + (2 + e + 2e^2 - e^3)C - 2 - 5e + 2e^2 + 3ES \\
\tilde{l}_{12} &= (1 - e^2)^{1/2}S(1 - C) \\
\tilde{l}_{21} &= (1 - e^2)^{1/2}[(1 + e)SC + (2 - e)S - 3EC] \\
\tilde{l}_{22} &= C^2 + (-1 - 2e + e^2)C + 1 \\
\tilde{l}_{33} &= (1 - e)^{3/2}(C - e)(1 - eC) \\
\tilde{m}_{11} &= (1 - e)^{1/2}S[-(1 + e)C + 2] \\
\tilde{m}_{12} &= (1 + e)^{1/2}[(2 - e)C^2 + 2(1 + e)C - 4 - e + 3ES] \\
\tilde{m}_{21} &= (1 + e)^{1/2}(1 - C)^2 \\
\tilde{m}_{22} &= (1 - e)^{1/2}[(2 + e + e^2)SC + 2S - 3(1 + e)EC] \\
\tilde{m}_{33} &= (1 - e)^{3/2}S(1 - eC) \\
\tilde{p}_{11} &= n(a/r)^2[(e + e^2 - e^3)SC^2 + (-2 - 5e + 2e^2)SC + (1 + e + 3e^2 - e^3)S + 3E(C - e)] \\
\tilde{p}_{12} &= n(a/r)^2(1 - e^2)^{1/2}[eC^3 - 2C^2 + C + 1 - e] \\
\tilde{p}_{21} &= n(a/r)^2(1 - e^2)^{1/2}[-(e + e^2)C^3 + (2 + 5e)C^2 - (1 + e)C - 1 - 3e + e^2 + 3ES] \\
\tilde{p}_{22} &= n(a/r)^2S[eC^2 - 2C + 1 + e - e^2] \\
\tilde{p}_{33} &= -n(1 - e)^{3/2}S \\
\tilde{q}_{11} &= n(a/r)^2(1 - e)^{1/2}[(e + e^2)C^3 - 2(1 + e)C^2 + 2C + 1 - e] \\
\tilde{q}_{12} &= n(a/r)^2(1 + e)^{1/2}[(2e - e^2)SC^2 - (4 + e)SC + (1 + e)^2S + 3E(C - e)] \\
\tilde{q}_{21} &= n(a/r)^2(1 + e)^{1/2}S[eC^2 - 2C + 2 - e] \\
\tilde{q}_{22} &= n(a/r)^2(1 - e)^{1/2}[-(2e + e^2 + e^3)C^3 + (4 + 5e + 5e^2)C^2 - (1 + 3e)C - 2 - 3e - e^2 + 3(1 + e)ES] \\
\tilde{q}_{33} &= n(1 - e)^{3/2}C
\end{aligned}
$$

Table 2 Hyperbolic reference orbit

$$
\begin{aligned}
ch &= \cosh F \qquad sh = \sinh F \\
\tilde{l}_{11} &= (-1 - e + e^2)ch^2 + (-2 - e - 2e^2 + e^3)ch + 2 + 5e - 2e^2 + 3Fsh \\
\tilde{l}_{12} &= (e^2 - 1)^{1/2}sh(ch - 1) \\
\tilde{l}_{21} &= (e^2 - 1)^{1/2}[(1 + e)shch + (2 - e)sh - 3Fch] \\
\tilde{l}_{22} &= ch^2 + (-1 - 2e + e^2)ch + 1 \\
\tilde{l}_{33} &= (e - 1)^{3/2}(e - ch)(ech - 1) \\
\tilde{m}_{11} &= (e - 1)^{1/2}sh[(1 + e)ch - 2] \\
\tilde{m}_{12} &= (e + 1)^{1/2}[(-2 + e)ch^2 - 2(1 + e)ch + 4 + e + 3Fsh] \\
\tilde{m}_{21} &= (e + 1)^{1/2}(ch - 1)^2 \\
\tilde{m}_{22} &= (e - 1)^{1/2}[(2 + e + e^2)shch + 2sh - 3(1 + e)Fch] \\
\tilde{m}_{33} &= (e - 1)^{3/2}sh(ech - 1) \\
\tilde{p}_{11} &= n_1(a/r)^2[(-e - e^2 + e^3)ch^2sh + (2 + 5e - 2e^2)chsh + (-1 - e - 3e^2 + e^3)sh + 3F(e - ch)] \\
\tilde{p}_{12} &= n_1(a/r)^2(e^2 - 1)^{1/2}[ech^3 - 2ch^2 + ch + 1 - e] \\
\tilde{p}_{21} &= n_1(a/r)^2(e^2 - 1)^{1/2}[(e + e^2)ch^3 - (2 + 5e)ch^2 + (1 + e)ch + 1 + 3e - e^2 + 3Fsh] \\
\tilde{p}_{22} &= n_1(a/r)^2sh[ech^2 - 2ch + 1 + e - e^2] \\
\tilde{p}_{33} &= -n_1(e - 1)^{3/2}sh \\
\tilde{q}_{11} &= n_1(a/r)^2(e - 1)^{1/2}[(e + e^2)ch^3 - 2(1 + e)ch^2 + 2ch + 1 - e] \\
\tilde{q}_{12} &= n_1(a/r)^2(e + 1)^{1/2}[(-2e + e^2)shch^2 + (4 + e)shch - (1 + e)^2sh + 3F(e - ch)] \\
\tilde{q}_{21} &= n_1(a/r)^2(e + 1)^{1/2}sh[ech^2 - 2ch + 2 - e] \\
\tilde{q}_{22} &= n_1(a/r)^2(e - 1)^{1/2}[(2e + e^2 + e^3)ch^3 - (4 + 5e + 5e^2)ch^2 + (1 + 3e)ch + 2 + 3e + e^2 + 3(1 + e)Fsh] \\
\tilde{q}_{33} &= n_1(e - 1)^{3/2}ch
\end{aligned}
$$

The right-hand side of (1) is part of the general solution, with nonzero $\mathbf{f}$; it is, strictly, the complementary function To complete the general solution, the particular integral must be added; this can be written

$$\text{P I} = \begin{bmatrix} \mathbf{L}(t,T) & \mathbf{M}(t,T) \\ \mathbf{P}(t,T) & \mathbf{Q}(t,T) \end{bmatrix} \int_{t_0}^{t} \begin{bmatrix} -\mathbf{M}^{\mathrm{T}}(\tau,T) \\ \mathbf{L}^{\mathrm{T}}(\tau,T) \end{bmatrix} \mathbf{f}(\tau)\, d\tau \quad (3)$$

Here $\mathbf{f}(\mathbf{r},t)$ is evaluated along the reference orbit, so that it becomes a known function of the time For the general solution (3) is added to the right-hand side of (1), and the latter is evaluated using (2) The result is the basic formula of the method; it follows from formulas given in Ref 1 The formulas are simplified if the axes of coordinates are chosen so that the x axis points toward the pericentron of the reference orbit and the y axis points toward that point for which the true anomaly is 90°; this system will be called the *orbital reference system* Only a rotation matrix is needed to reach any other system that may be more fundamental; this and the matrix components are simple functions of the elements of the reference Keplerian orbit

The accuracy of the formulas depends on the magnitude of the residuals whose squares and products have been neglected It has been found to be a reliable test to compare $(\delta\mathbf{r})^2$ with the greatest tolerable error For instance, if six places of decimals are required, then we should have $(\delta\mathbf{r})^2 < 5\,10^{-7}$ When this fails to be the case a new and up-to-date reference orbit must be chosen The residual in position, rather than that in velocity, is used for the test because $\mathbf{f}$ depends normally on position only The accuracy required of the calculation is therefore seen to be an input parameter

The calculation of the perturbations does not involve the numerical solution of a differential equation, in the normal sense, but merely some method of numerical quadrature (the author used the repeated Simpson rule); here round-off errors are less trouble and are fairly easily kept in hand The accuracy of the integration is, in fact, another input parameter

At the start, the data are position and velocity at some time t_0 The best that can be done for a reference orbit is to pick the osculating orbit at t_0, so that $\delta\mathbf{r}_0 = \delta\mathbf{r}_0' = 0$ This initial osculating orbit will serve as a reference until $(\delta\mathbf{r})^2$ becomes too large, at t_1, say When this happens two courses are open It is possible to go back to an instant when $(\delta\mathbf{r})^2$ was not too large, find the position and velocity in the true orbit, and go on from there with a new osculating orbit as reference Or it is possible to use the knowledge gained of the true orbit and devise a Keplerian orbit that will not be osculating at any instant, but will adequately represent the true orbit over a longer period of time This will be a *mean orbit* Mean orbits are important because if the true orbit is to be divided up into arcs, the fewer the better Infinite variety is possible in choosing mean orbits; the method used by the author is to make the orbit pass through the *positions*

in the true orbit at times t_0 and t_1 Again, position rather than velocity is used because **f** does not normally involve the velocity This choice of a mean orbit is probably the simplest of any to program, and it has been found in practice that it is nearly always substantially more effective than the initial osculating orbit preceding it

Having chosen the mean orbit, it is necessary to start again from t_0; but $\delta\mathbf{r}_0'$ will not be zero Next, suppose that t_2 is the latest time found for which $(\delta\mathbf{r})^2$ is small enough It has been found best to choose a fresh initial osculating orbit starting at t_2 (so that the arc from t_0 to t_2 is now completed) and go on from this time, repeating the process just described

3 Detailed Formulas

Formulas for the components of the matrices in (2) are given in Ref 2; but (3) is not yet in a form suitable for its practical application The integrand contains six elements, and the numerical evaluation of the integral involves calculating the integrand at equal intervals of the independent variable Using the repeated Simpson rule, the accuracy is checked by insuring that the maximum fourth difference of these is less than some quantity (decisions on changing the interval are also based on this test) For efficiency, all six elements in the integrand should therefore be of the same order of magnitude; but, as they stand, they do not even have the same dimensions **M** has the dimension of the time, and **L** is dimensionless, but let n be the mean motion in the reference orbit The integral can be written as

$$\frac{1}{n}\begin{bmatrix} L & nM \\ P & nQ \end{bmatrix}\int_{t_0}^{t}\begin{bmatrix} -nM^{\mathrm{T}} \\ L^{\mathrm{T}} \end{bmatrix} f(\tau)\, d\tau$$

and since n has the dimension (time)$^{-1}$, that difficulty is resolved

More serious, if highly eccentric orbits are to be considered, is the fact that $(1 - e)$ appears in the denominators of several matrix components This can be removed by some algebraic manipulation The details will not be given here, but only the results

Finally, the time is not a suitable independent variable for the integration The matrix components can be expressed simply, in closed form, in terms of the eccentric or true anomalies, but not in terms of the time So with the time as the independent variable, at each step Kepler's equation must be solved the hard way, with its concomitant iterations Also, in an eccentric orbit the time intervals for the numerical integration will be relatively short near pericentron and relatively long near apocentron; the interval can be changed automatically, but the fewer changes the better Similarly, if the true anomaly were the independent variable, the points would be relatively closely spaced near apocentron, and there would be trouble in the case of a hyperbolic orbit The ideal choice is the eccentric anomaly; this keeps the change of interval down to a minimum, and Kepler's equation only has to be solved for the time, given the eccentric anomaly Let E be the eccentric anomaly, a the semimajor axis, and r the radius vector, all in the reference orbit; then the change of variable is accomplished by $dt = (r/na)dE$ With a little manipulation, r can be absorbed into the matrix components

The choice of the eccentric anomaly as the independent variable may seem awkward at first sight, since a frequent aim of an orbital calculation is the production of an ephemeris over regular intervals of time In fact, the calculation of such an ephemeris was incorporated into the program without prohibitive complication But in practice, what is needed is the ability to calculate position and velocity at arbitrary times This is conventionally done by interpolation, based on values tabulated at regular intervals of time, and it would be possible to replace the time by the eccentric anomaly in a reference orbit at the cost of solving Kepler's equation But the author recommends the use of the Taylor series, expanded about the tabulated values These can be made to include the effects of all the disturbing forces, and, normally, to require the availability of fewer tabulated values than are needed for the conventional interpolation Also, there is no need for these values to be tabulated regularly with respect to any variable

Let

$$\tilde{\mathbf{L}}(E) = \begin{bmatrix} \tilde{l}_{11}(E) & \tilde{l}_{12}(E) & 0 \\ \tilde{l}_{21}(E) & \tilde{l}_{22}(E) & 0 \\ 0 & 0 & \tilde{l}_{33}(E) \end{bmatrix}$$

with $\tilde{\mathbf{M}}$, $\tilde{\mathbf{P}}$, and $\tilde{\mathbf{Q}}$ defined similarly The basic formula can be written as follows:

$$\begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{r}' \end{bmatrix} = (a^2/nrr_0)(1 - e)^{-3/2}\begin{bmatrix} \tilde{\mathbf{L}}(E) & \tilde{\mathbf{M}}(E) \\ \tilde{\mathbf{P}}(E) & \tilde{\mathbf{Q}}(E) \end{bmatrix} \times \begin{bmatrix} -\tilde{\mathbf{M}}^{\mathrm{T}}(E_0) \\ \tilde{\mathbf{L}}^{\mathrm{T}}(E_0) \end{bmatrix}\delta\mathbf{r}_0' + (a/n^2r)(1 - e)^{-3/2} \times \begin{bmatrix} \tilde{\mathbf{L}}(E) & \tilde{\mathbf{M}}(E) \\ \tilde{\mathbf{P}}(E) & \tilde{\mathbf{Q}}(E) \end{bmatrix}\int_{E_0}^{E}\begin{bmatrix} -\tilde{\mathbf{M}}^{\mathrm{T}}(\eta) \\ \tilde{\mathbf{L}}^{\mathrm{T}}(\eta) \end{bmatrix}\mathbf{f}(\eta)\, d\eta \quad (4)$$

This formula applies directly to elliptic reference orbits, where $n^2a^3 = \mu$, and $r = a(1 - e\cos E)$ For hyperbolic orbits a is negative and should be replaced in the forementioned by $-a$ n should be replaced by n_1, where $-n_1^2a^3 = \mu$, and $r = a(1 - e\cosh F)$, F being the hyperbolic eccentric anomaly Also $(1 - e)$ should be replaced by $(e - 1)$ Otherwise separate formulas must be used for the matrix components They are listed in Tables 1 and 2

To find the mean orbit (as just defined) it would be sufficient to use any method for finding a Keplerian orbit going through $\mathbf{r}_0$ at t_0 and $\mathbf{r}_1$ at t_1 But the difference between the initial osculating and the mean orbits is a small residual in velocity at t_0 Let this be $d\mathbf{r}_0'$, and let $\delta\mathbf{r}_1$ and $\delta\mathbf{r}_1'$ be the residuals at t_1 found from the integration with respect to the initial osculating orbit Then

$$\begin{bmatrix} \delta\mathbf{r}_1 \\ \delta\mathbf{r}_1' \end{bmatrix} = \begin{bmatrix} \mathbf{P}(t_1,t_0) & \mathbf{M}(t_1,t_0) \\ \mathbf{L}(t_1,t_0) & \mathbf{Q}(t_1,t_0) \end{bmatrix}\begin{bmatrix} 0 \\ d\mathbf{r}_0' \end{bmatrix}$$

so that

$$d\mathbf{r}_0' = \mathbf{M}^{-1}(t_1,t_0)\delta\mathbf{r}_1 \quad (5)$$

$\mathbf{M}(t_1,t_0)$ can be found from matrices already calculated, using relations in (2) $d\mathbf{r}_0'$ takes us from the true orbit at t_0 to the mean orbit; when integrating with respect to the mean orbit, the value of $\delta\mathbf{r}_0'$ in (1) or (4) will be the negative value of $d\mathbf{r}_0'$ because $\delta\mathbf{r}_0'$ must take us from the mean orbit to the true orbit At each change of reference orbit, the orbital reference system will have to be changed

In practice it was found necessary, for different reference orbits, to vary the restraints on accuracy (the comparison with $(\delta\mathbf{r})^2$ and the accuracy required of the numerical quadrature) for two main reasons If an initial osculating orbit is to be followed by a mean orbit, the former need not be as accurate as the latter, since the ground is to be covered again Also there is a chance that the mean orbit may be more effective if the points defining it are more widely spaced (But a mean orbit can cause trouble by failing before the preceding initial osculating orbit failed; this is apt to happen when the starting point is near pericentron in an eccentric orbit) Moreover, the accuracy required of the integral is governed by the factor $(1 - e)^{-3/2}$ If the eccentricity of the reference orbit is to change by much (and this happens in a circumlunar orbit when the origin is changed), then the accuracy required of the quadrature will vary also

4 Analysis of the Orbit

When an orbit has been calculated there will be information about position and velocity where this may be wanted, but there also will be provided the position and velocity defining each reference orbit; all of this information would be referred to some fundamental reference system Suppose that some analysis concerns the times t_a,t_b; then the matrizant $\Omega(t_a,t_b)$ will be needed If these times lie in the same reference arc, then (2) can be used immediately to give the matrizant If not, suppose that t_i starts the arc containing t_a, t_{i+k} starts the arc containing t_b, and t_{i+1}, , t_{i+k-1} start intermediate arcs Then

$$\Omega(t_b,t_a) = \Omega(t_b,t_{i+k})\Omega(t_{i+k},t_{i+k-1}) \Omega(t_{k+2},t_{k+1})\Omega(t_{k+1},t_a) \quad (6)$$

Each matrizant is calculated using the parameters corresponding to the time that begins the appropriate reference orbit Equation (6) holds even if different arcs are referred to different origins, provided that the orientation of the axes remains the same in space

It also is possible fairly simply to investigate the effects of forces that may have been neglected If these are substituted for $\mathbf{f}$ in (3), and the parameters for the matrix components used in the evaluation of the integral are those for the reference orbits already found, then the evaluation of (3) will provide a history of the effects that these forces would have had if they had been included in the calculation of the orbit (always assuming these effects to be small) The same applies to small changes in constants used in the work, such as the mass of the moon A small change in a constant will lead to a perturbing force slightly different from the one used in the calculation of the orbit; if this difference is substituted for $\mathbf{f}$ in (3) the effects of the change can be calculated It may also be possible to investigate unknown forces causing deviation from the calculated motion, if that deviation is known well enough, for then (3) would be treated as an integral equation for the unknown $\mathbf{f}$

References

[1] Danby, J M A, "Integration of the equations of planetary motion in rectangular coordinates," Astron J 67, 287–299 (1962)
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JANUARY 1964 AIAA JOURNAL VOL 2, NO 1

The Matrizant of Keplerian Motion

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This paper contains a documentation of various forms of the matrizant of the different types of Keplerian motion (The *matrizant* is the matrix that relates residuals, or small departures from some known orbit, at different times) The formulas can be used in the analysis of Keplerian or nearly Keplerian orbits when relations between coordinates at different times are required

1 Introduction and Notation

TRADITIONALLY the methods of differential correction in celestial mechanics have been used to relate residuals between observed and computed quantities to small changes in the geometrical elements of reference Keplerian orbits These methods cannot be immediately applied when the relations between residuals in position and velocity at two different times are required, and since there is some contemporary preoccupation with these quantities (the geometrical elements being incidental), the author feels that the following forms of the matrizant may prove to be useful Expressions for the matrizants of particular kinds of orbits have been published, but so far as the author is aware, no general survey has appeared in the literature

Keplerian motion is the motion of a particle subject to the force function μ/r Let us assume that some reference Keplerian orbit is completely defined, so that position $\mathbf{r}$ and velocity $\mathbf{r}'$ are known for any time In a "neighboring" Keplerian orbit, position and velocity will be $\mathbf{r} + \delta\mathbf{r}$ and $\mathbf{r}' + \delta\mathbf{r}'$, and the relation between these residuals at times t_0 and t will be given (to the first order in these quantities) by a formula of the kind

$$\begin{bmatrix}\delta\mathbf{r}\\ \delta\mathbf{r}'\end{bmatrix} = \Omega(t,t_0)\begin{bmatrix}\delta\mathbf{r}_0\\ \delta\mathbf{r}_0'\end{bmatrix} \quad (1)$$

It is assumed that $\delta\mathbf{r}$ is the column matrix

$$\begin{bmatrix}\delta x\\ \delta y\\ \delta z\end{bmatrix}, \text{ etc}$$

There are currently many names for the matrix $\Omega(t,t_0)$ The author prefers "matrizant," as having historical precedence through the work of Peano and Baker (see, for instance, Ref 1) It is convenient to subdivide $\Omega(t,t_0)$ into four three-by-three matrices, for example,

$$\Omega(t,t_0) = \begin{bmatrix}\mathbf{L}(t,t_0) & \mathbf{M}(t,t_0)\\ \mathbf{P}(t,t_0) & \mathbf{Q}(t,t_0)\end{bmatrix} \quad (2)$$

In 1932 Bower[2] published a method of differential corrections that, with a few modifications, can be used to compute the matrizant The method applies to elliptic orbits but can be easily broadened to cover hyperbolic orbits also The orientation of the axes is arbitrary However, the components of the matrizant are not given directly but are found from a series of intermediate functions The method will be described below in relation to nearly circular orbits, but most of this paper will deal with explicit formulas

Received May 28, 1963; revision received October 16, 1963 This work was supported by the Minneapolis-Honeywell Regulator Company, Aeronautical Division

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